PERIOD–III

M12CH8

TOPIC

8

Logical Reasoning

8.1. STATEMENTS

We all know that human beings can think more logically as compared to other species, i.e., animals or birds. This ability makes them far more superior to other species. Logic as language of Mathematics is the study of general pattern of reasoning. In this unit, we shall study some basics of logical reasoning.

We express our ideas by means of sentences. A sentence can be (i) true (ii) false (iii) both.

Now, consider the following sentences:

(i) 5 is less than 7

(ii) 5 is greater than 8

(iii) Mathematics is an interesting subject.

Without any confusion, we can clearly decide that (i) is true and (ii) is false. What about (iii)? Some students may agree to it and call it true while others may disagree and call it false. The sentence is ambiguous. We cannot say whether it is always true or false.

A sentence which is either true or false but not both is called a 'logical statement' or 'a mathematically acceptable statement' or briefly 'a statement'.

In the above example, (i) and (ii) are statements whereas (iii) is not a statement.

In logical reasoning, the basic unit involved is a statement.

A true sentence is also called a **valid statement** and a false sentence is also called an **invalid statement**. A sentence which is both true and false simultaneously is called a **paradox**. Every statement is a sentence, true or false, but every sentence need not be a statement. Statements are denoted by small letters *p*, *q*, *r*, … .

For example, p : 7 is a prime number

$q : \sqrt{2}$ is a rational number

In the grammatical sense, a statement is a declarative or assertive sentence. It is neither imperative nor interogative, nor optative, nor exclamatory.

For example, consider the following sentences:

(a) (i) 2012 was a leap year.

(ii) π is an irrational number.

(iii) Every square is a rectangle.

Each of these sentences is a declaration or an assertion. Each of them is true and, hence, each of them is a valid statement.

(b) (i) All prime numbers are odd.

(ii) $\sqrt{2}$ is a rational number.

(iii) Every set is a finite set.

Each of these sentences is a declaration or an assertion. Each of them is false and, hence, each of them is an invalid statement.

(c) (i) Switch on the light.

(ii) Please open the door.

(iii) Get out.

Each of the above sentences is a command or a request and, hence, an imperative sentence. None of them can be called true or false. Hence, none of them is a statement.

(d) (i) How do you do?

(ii) Where are you going?

(iii) When will you wake up?

Each of the above sentences is a question and, hence, an interogative sentence. None of them can be called true or false. Hence, none of them is a statement.

(e) (i) May you live long!

(ii) May God bless you!

(iii) Good morning every body!

Each of the above sentences is a wish or desire and, hence, an optative sentence. None of them can be called true or false. Hence none of them is a statement.

(f) (i) How beautiful is the rainbow!

(ii) Hurrah! I have passed with distinction.

(iii) What a fragrance!

Each of the above sentences is exclamatory. None of them can be called true or false. Hence, none of them is a statement.

8.2. NEGATION OF STATEMENTS

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$. It is read as 'not p'.

Let p : 7 is a prime number

be a given statement. Now consider the following statements:

(i) 7 is not a prime number.

(ii) It is false that 7 is a prime number.

(iii) It is not true that 7 is a prime number.

(iv) It is not the case that 7 is a prime number.

Each of these statements is opposite in meaning to the given statement. Hence each of these statements is the negation of p, i.e., each of these statements is $\sim p$.

Thus, negation of a statement p is formed by inserting the word 'not' if absent and by dropping the word 'not' if present. The negation of p is also formed by writting 'It is false that' or 'It is not true that' or It is not the case that' before p.

The truth value of $\sim p$ is always the opposite of the truth value of p. Negation changes a true statement into a false statement and a false statement into a true statement. In other words, if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

Example 1. *Negate the following statements:*

(i) *Kofi is not a lazy boy.* (ii) $\sqrt{7}$ *is a rational number.*

Solution. (i) Let p : Kofi is not a lazy boy.

Then $\sim p$: Kofi is a lazy boy.

(obtained by dropping 'not')

Note: The negation of p may also be written as:

$\sim p$: **It is false that** Kofi is $\underbrace{\text{not a lazy boy}}_{p}$.

or $\sim p$: **It is not the case that** Kofi is $\underbrace{\text{not a lazy boy}}_{p}$.

(ii) Let p : $\sqrt{7}$ is a rational number.

Then ~ p : $\sqrt{7}$ is not a rational number (obtained by inserting 'not')

Note: The negation of p may also be written as:
~ p : It is false to say that $\sqrt{7}$ is a rational number.
or ~ p : It is not the case that $\sqrt{7}$ is a rational number.
or ~ p : $\sqrt{7}$ is an irrational (not rational)number.
Clearly, p is false and ~ p is true.

8.3. IMPLICATIONS ($\Rightarrow$, $\Leftrightarrow$)

(a) 'If-then' Implication or Conditional Statement

In Mathematics and in our day-to-day life, expressions of the form 'if p', then q' occur very often.

For example:

(i) If $x = 4$, then $x^2 = 16$
(ii) If $3x - 2 = 10$, then $x = 4$
(iii) If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.
(iv) If it rains, then I will not go out for a walk.

If p and q are two statements, then the statement 'if p, then q' is called an if, then' implication or simply an **implication or a conditional statement**. It is denoted by $p \Rightarrow q$ and read as 'p implies q.' Here p is called the **antecedent or hypothesis** and q is called the **consequent or conclusion**.

In example (i) above, let $p : x = 4$, $q : x^2 = 16$
then the symbolic form of statement (i) is $p \Rightarrow q$

In example (ii) above, let $p : 3x - 2 = 10$, $q : x = 4$
then the symbolic form of statement (ii) is $p \Rightarrow q$. Similarly, for (iii) and (iv)

It is very important to observe that if p is true, then q must be true, i.e., whenever the hypothesis holds, the conclusion must hold. However, when p is false, then q may be true or false, i.e., no restriction on q.

Consider p : You are born in some country

q : You are a citizen of that country

then $p \Rightarrow q$ is the statement:

If you are born in some country, then you are a citizen of that country.

Clearly, if p is true then q is true. What happens when p is false? i.e., when you are not born in a county, then you are not a born citizen yet you can acquire citizenship.

It may be noted that '$\Rightarrow$' is **not commutative**. Thus $p \Rightarrow q$ is different from $q \Rightarrow p$.

$p \Rightarrow q$ means p is sufficient for q

while $q \Rightarrow p$ means p is necessary for q.

(b) 'If and only if' Implication or Double Implication or Biconditional Statement

Let p and q be two statements such that $p \Rightarrow q$ and $q \Rightarrow p$ i.e., 'if p then q' and 'if q then p', then this compound statement is called 'if and only if' implication or double implication or biconditional statement. It is denoted by $p \Leftrightarrow q$ and read as 'p if and only if q'. For brevity, if and only if is written as 'iff'. Thus, $p \Leftrightarrow q$ or p iff q is a double implication or a biconditional statement.

Note that '$\Leftrightarrow$' is commutative. Thus, $p \Leftrightarrow q$ is same as $q \Leftrightarrow p$. Therefore, $p \Leftrightarrow q$ means p is necessary and sufficient for q.

For example:

(i) Since $3x - 2 = 10 \Rightarrow x = 4$

and $x = 4 \Rightarrow 3x - 2 = 10$

we say $3x - 2 = 10 \Leftrightarrow x = 4$

(ii) Let p : a triangle is equilateral

q : a triangle is equiangular

then $p \Rightarrow q$ is the statement 'If a triangle is equilateral then it is equiangular.

$q \Rightarrow p$ is the statement 'If a triangle is equiangular then it is equilateral'.

$p \Leftrightarrow q$ is the statement 'A triangle is equilateral, if and only if it is equiangular.'

8.4. USE OF VENN DIAGRAMS IN TESTING THE VALIDITY OF IMPLICATIONS

An **argument** is an assertion that a statement S follows from certain other statements $S_1, S_2, \ldots, S_n$.

The statement S is called the **Conclusion** and the statements $S_1, S_2, \ldots, S_n$ are called **hypothesis** or **premisis**.

An argument consisting of hypothesis $S_1, S_2, \ldots, S_n$ and conclusion S is said to be **valid** if S is true whenever all $S_1, S_2, \ldots, S_n$ are true, i.e., if $S_1, S_2, \ldots, S_n$ are all true $\Rightarrow$ S is true.

The validity of an argument can be tested by using Venn diagrams as follows:

(i) Represent the truth of the hypothesis by Venn diagrams.

(ii) Analyse the Venn diagrams to see whether they necessarily represent the truth of the conclusion. If so, then the argument is valid, otherwise it is invalid.

Example 2. *Use Venn diagrams to examine the validity of the following arguments:*

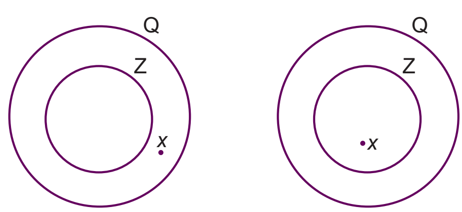
(i) S_1 : *All integers are rational numbers.*
S_2 : *x is a rational number.*
S : *x is an integer.*

(ii) S_1 : *All integers are rational numbers.*
S_2 : *x is a rational number.*
S : *x is not an integer.*

(iii) S_1 : *All integers are rational numbers.*
S_2 : *x is not a rational number.*
S : *x is not an integer.*

Solution. The three given statements in each part constitute an argument in which S_1, and S_2, are hypothesis and S is the conclusion.

Let Q denote the set of all rational numbers and let Z denotes the set of all integers. The truth of the statement S_1, $Z \subset Q$, is represented by placing the set Z entirely inside the set Q. The truth of the statement S_2 is represented by placing a dot labelled x inside the set Q. But the position of the dot with respect to the set Z is not known.



(a) (b)

x may be $\frac{2}{3}$ which is not an integer and hence, as shown in Fig. 1(a) the dot is outside the set Z.

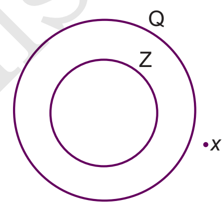
x may be –5 which is an integer and hence, as shown in Fig. 1 (b), the dot is inside the set Z.

Both the positions of x represent the truth of the statement S_2.

(i) The truth of the conclusion S that x is an integer does not necessarily follow from the truth of the hypothesis S_1 and S_2 in view of Fig. 1 (a). Hence the argument is invalid.

(ii) The truth of the conclusion S that x is not an integer does not necessarily follow from the truth of the hypothesis S_1 and S_2 in view of Fig. 1(b). Hence the argument is invalid.

(iii) Here the truth of the statement S_2 that x is not a rational number is represented by placing a dot labelled x outside the set Q as shown in Fig. Since the dot x is outside the set Q, it is necessarily outside the set Z of all integers. Therefore, x is not an integer.



(c)

Thus, the truth of the conclusion S follows from the truth of the hypothesis S_1 and S_2. Hence the argument is valid.

8.5. EQUIVALENT IMPLICATION

If the two statements p and q are such that $p \Rightarrow q$ is true and the converse statement $q \Rightarrow p$ is also true, then $p \Rightarrow q$ are **equivalent** if and only if both $p \Rightarrow q$ and its converse $q \Rightarrow p$ both true.

For example, $3x - 2 = 10 \quad \Rightarrow \quad x = 4$ and $x = 4 \quad \Rightarrow \quad 3x - 2 = 10$ are both true.

Therefore, $3x - 2 = 10 \quad \Rightarrow \quad x = 4$.

For any two statements p and q, $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$ (but not $\sim p \Rightarrow \sim q$ or $q \Rightarrow p$).

The symbol '$\Leftrightarrow$' is the equivalent implication sign.

$p \Leftrightarrow q$ is read as 'p' is equivalent to q'.

Note this carefully:

If $p \Rightarrow q$, then we can write the equivalent statement $\sim q \Rightarrow \sim p$.

8.6. VALID ARGUMENTS

An argument is **valid** if and only if the conclusion follows from other statement (the premises). The premise is the gives statement from which other conclusions can be drawn.

Note that the premise of an argument is always assumed to be true. Note also that the truth of the conclusion is irrelevent when testing for the validity of an argument. The fact that the conclusion is true is not sufficient for an argument to be valid.

Consider the following argument:

Monrovia is in Liberia

Liberia is in West Africa

Therefore Monrovia is in West Africa

The first two statements are the premises and the last statement is the conclusion

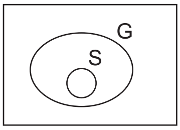
Note that although the two premises are false they are assumed to be true.

Note also that the conclusion is false although the argument is valid.

Example 3. Consider the statement:

q : Therefore is **no** soldier who does **not** use gun.

It means that *all* soldiers use gun. That is the set S = {soldiers} is a *subset* of the set G = {people who use gun}.



S = G

The statement can be represented in a Venn diagram as shown in figure.

Similarly, the statement:

"q : All policemen wear uniform" means

P = {people} is a *subset* of

U = {people who wear uniform}

Also, the statement:

"p : If students work hard **then** they will pass their examinations" means

S = {students who work hard} is a *subset* of {students who pass their examinations}

Example 4. *Consider the following statement:*

p : All soldiers are men

q : There is no sodier who does not use gun.

(*i*) *If M = {men}, S = {solidier) and G = {people with gun} draw a Venn diagram to illustrate p and q.*

(*ii*) *Statement whether or not each of the following is a valid conclusion from p and q.*

(*a*) *Men who use gun are soldiers*

(*b*) *All men use gun*

(*c*) *Men who do not use gun are not soldiers.*

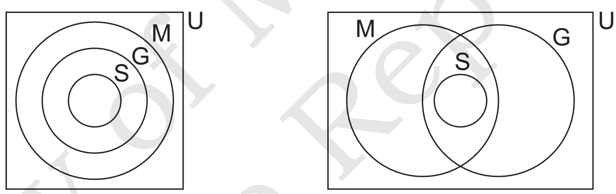
Solution. (*i*) Let U = {all people}, M = {Men}, S = {soldiers}

and G = {people with gun}

From stateents p and q we can write the following:

(1) $S \subset M$ and (2) $S \subset G$.

Figure (i) and (ii) are the possible diagrams illustrating the two statement p and q.



(i) (ii)

(ii) (a) Men who use gun are within region G but not all those who use gun are soldiers since a man can be inside region G but outside region S. Therefore the statement is not always true and not a valid deduction from p and q.

(b) All men are within region M but outside region G. Therefore the statements is not always true and not a valid deduction from p and q.

(c) Men who do not use gun are outside region G and are also outside region S. Therefore the statement is a valid deduction from p and q.

Example 5. *Consider the following statements.*

X : All soldiers are hardworking.

Y : No hardworking person is careless.

Draw a Venn diagram to illustrate the above statements.

Which of the following are valid conclusions from the statements X and Y?

(*i*) *Kwasi is a student* $\Rightarrow$ *Kwasi is not careless.*

(*ii*) *Asiedu is hardworking* $\Rightarrow$ *Asiedu is a student.*

(*iii*) *Efua is careless* $\Rightarrow$ *Efua is not a student.*

Solution. The statements are about students, hardworking persons, and careless persons, therefore;

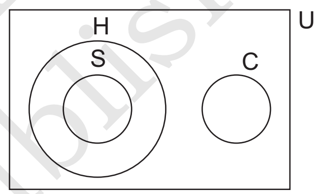
Let U = {all persons}, S = {hardworking persons} and

C = {Careless persons}

From the statements X and Y, we can write the following:

1. $S \subset H$ and H and C are disjoint sets.

Figure is the Venn diagram illustrating the statement X and Y.



(i) Kwasi is a student means he is in region S and therefore cannot be in region C. Therefore the statement

"Kwasi is a student $\Rightarrow$ Kwasi is not careless is valid.

(ii) Asiedu is hardworking means he is in region H and therefore can be within region S or outside S. Therefore the statement is not always true and not a valid conclusion.

(iii) Efua is careless means she is in region C and therefore cannot be in region S. Therefore the statement is always true and valid.

Using the fact that if $p \Rightarrow q$, then $\sim q \Rightarrow \sim p$.

We can also deduce the validity of an argument using the fact that; if $p \Rightarrow q$ is true then the equivalent statement $\sim q \Rightarrow \sim q$ is also true.

Example 6. *Consider the following statements.*

p : Abena has measles

q : Abena is in the hospital

If p $\Rightarrow$ q, state whether or not the following statement are valid.

(*i*) *If Abena is in the hospital, then she has measles.*

(*ii*) *If Abena is not in the hospital, then she does not have measles.*

(*iii*) *If Abena does not have measles, then she is not in the hospital.*

Solution. $p \Rightarrow q$ means q is true only if p is true.

We can also write the equivalent statement $\sim q \Rightarrow \sim p$

where $\sim p$ and $\sim q$ are the negations of the statements p and q respectively.

Note: We cannot write $q \Rightarrow p$ and $\sim p \Rightarrow \sim q$

(i) If Abena is in the hospital, then she has measles means $q \Rightarrow p$. Therefore the statement is not valid.

(ii) If Abena is not in the hospital, then she does not have measles means $\sim q \Rightarrow \sim p$. Therefore the statement is valid.

(iii) If Abena does not have measles, then she is not in the hospital means $\sim p \Rightarrow \sim q$. Therefore the statement is not valid.

Note that if $p \Rightarrow q$ we can write the equivalent statement $\sim p \Rightarrow \sim q$. But not $\sim p \Rightarrow \sim q$ or $q \Rightarrow p$.

Using the chain rule

The chain rule states that:

If p, q and r are any three statements

such that: $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$

Example 7. *Determine whether or not the following argument is valid:*

Monrovia is Liberia

Liberia is in West Africa

Therefore Monrovia is in West Africa

Solution. Let p : {Y is in Monrovia }

q : {Y is in Liberia} and

r : {Y is in West Africa}

The first premise means $p \Rightarrow q$ and the second premise means $q \Rightarrow r$

Hence by the chain rule $p \Rightarrow r$ *i.e.*, Monrovia is in West Africa

Therefore the conclusion follows from the premises and the argument is valid.

EXERCISE

1. Sentences involving variable time such as 'today' 'tomorrow' or 'yesterday' are not statements. "Tomorrow is Wednesday".
2. Sentences involving variable places such as 'here' 'there' are also not statements. "London is far from here".
3. Sentences involving pronouns time such as 'he' 'she' 'they' are not statements. 'He is a doctor'.
4. Use Venn diagrams to examine the validity of the following argument:

   S_1: If a man is a bachelor, he is unhappy.

   S_2: If a man is unhappy, he dies young.

   S_3: If All bachelors die young.

5. Consider the following statement:

X: *All* junior secondary pupils wear unfirom.

Y: *Most* junior secondary pupils are well behaved.

(*a*) Draw a venn diagram to illustrate the above statement.

(*b*) Using the venn diagram or otherwise determine which of the following implications are valid deductions from X and Y.

(*i*) Osei wears uniform $\Rightarrow$ Osei is a junior secondary pupil

(*ii*) Kofi is junior secondary pupils $\Rightarrow$ he is well behaved

(*iii*) Kwasi does not wear uniform $\Rightarrow$ he is not a junior secondary pupil

6. The following statements are true of a certain society:

p: *All* friends are intelligent

q: *No* intelligent person is conservative

(*a*) Draw a venn diagram to illustrate the above statement.

(*b*) Using your Venn diagram, complete the following statements

(*i*) Kweku who is not my friend _______

(*ii*) Ama who is not conservative _______

(*iii*) John who is intelligent _______

7. Consider the following statements:

X: *All* students with measles stay in the sick bay.

Y: *All* students in the sick bay *do not* do homework.

Which of the following statements is/are valid conclusion from the statements X and Y.

(*i*) Kofi does not have measles so kofi does his homework.

(*ii*) George has done his homework therefore he does not stay in the sick bay.

(*iii*) Jane does not have measles so she does not stay in the sick bay

8. Consider the following statements:

p: Kweku trains hard

q: Kweku wins the race

If $p \Rightarrow q$ which of the following statements are valid?

(*i*) If Kweku wins the race then he has trained hard

(*ii*) If Kweku does not train hard then he will not win the race

(*iii*) If Kweku does not win the race then he has not trained hard.

9. Consider the following statements:

p: Kwesi trains hard

q: Kwesi is rich

If $p \Rightarrow q$ which of the following statements are valid?

(*i*) If Kwesi is rich then works hard.

(*ii*) If Kwesi does not work hard the he is not rich.

(*iii*) If Kwesi does not work hard then he will not be rich.